Students' Ability to Confine Their Application of Knowledge: The Case of Mathematics and Science

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Introduction

Research in science and mathematics education has indicated that students sometimes produce mutually incompatible solutions to essentially similar problems. For instance, they provide uncongenial responses to the same problem when it is given in two different contexts (Clough & Driver, 1986; Hiebert & Lefevre, 1986). This observation raises issues which are of great theoretical and practical importance to education. A central issue is, how do various factors affect the student's choice of response to a given problem? More specifically, how do factors related to the problem, i.e. its structure, the numerical data, the figural aspects, and the content domain in which it is embedded, affect the student's solution? What effects do factors related to the solver, i.e. age, grade level, and instruction, have on his or her solution to a given problem?

These issues are usually approached by presenting students with a variety of essentially similar problems and investigating the relationship between the specific features of each of these problems and students' responses (Silver, 1986; Stavy, 1990). A less conventional way is examining students' responses to problems which are externally similar though essentially different and require different solutions.

Two essentially different problems, a mathematical problem related to potential infinity and a scientific problem related to the particulate nature of matter, were chosen for this study. These two problems are figurally and spatially similar but since each stems from an entirely different theoretical framework, they require different responses. The main aims were: (a) to determine if students tend to erroneously produce the same response to both problems or rather give different, adequate responses to each of them; (b) to investigate whether students' responses change with formal, school-based instruction; and (c) to assess the effects of exposing students to the adequate interpretations to each of these problems.

Method

Subjects

Two-hundred upper middle-class students from the Sharon area in Israel participated in this study. Fifty students were randomly selected from the seventh, eighth, tenth, and twelfth grade levels in the same school. The tenth- and twelfth-grade students studied mathematics as their major subject.

All participating students studied mathematics and science according to the national curriculum. The topics that the subjects studied are as follows: In the seventh grade they did not receive any instruction in mathematics concerning geometry or infinite processes. At the time of the research, they had finished studying a chapter on the particulate nature of matter. The eighth-grade students also had not received any instruction concerning geometry or infinite processes. In that year, they had received formal instruction in science related to elements, compounds, and the periodic table. The tenth-grade students had studied basic Euclidean geometry (i.e., undefined and defined terms, axioms, postulates, definitions, theorems, and proofs) in the ninth and tenth grades. During these years, they did not receive any additional instruction in science concerning the structure of matter. The twelfth-grade students had studied, in the eleventh and twelfth grades, an introductory course in calculus in which they dealt with infinite series, limits, and integrals. They also studied, in these grades, science on a minor level (stoichiometry, the structure of the atom, acids and bases, oxidation and reduction, etc.).

The students were taught by several teachers, all of whom had taught in more than one grade level. All teachers had at least a bachelor's degree in either mathematics or science, a teacher certificate for secondary school, and at least three years of teaching experience.

The Problems

Initially, the following two problems were presented:

The successive division of a line segment
Consider a line segment AB. Divide AB into halves. Divide each of the obtained line segments again in halves. Continue halving the obtained line segments in the same way. Will this process come to an end? Explain your answer.

The successive division of a copper wire.
Consider a piece of copper wire. Divide it into halves. Divide each of the obtained parts again in halves. Continue halving the obtained parts in the same way. Will this process come to an end? Explain your answer.

These two problems are fundamentally different. In the first, an ideal, geometrical segment is considered whereas the second deals with a material, copper wire. The adequate responses to these two problems are, in the case of the line segment, the halving process is endless, whereas in the case of the copper wire, the halving process stops upon reaching the atomic level. The external similarity of the problems may encourage students to give the same response.

The Intervention

To assess the effect of presenting the student with the clues for adequate solutions to each of the above mentioned problems, the subjects were exposed to the following question:

> *A student named Karen asked the following question, "I understand that when dividing a copper wire time and time again, the process will end when reaching the atom level. In contrast, the successive division of a line segment is an endless process. Why is it so?" In your opinion, do you think that the above statement made by Karen is correct? Why?*

Students' responses to this question provided information about changes in their answers to the two problems as a result of the exposure to the adequate interpretations.

Procedure

The two problems and the intervention task were administered to all students during one class period (about 45 minutes) in the first week of March, 1989. Half of the students in each grade level received the mathematical problem first while the other half was presented with the problems in reversed order. The effect of the order of presentation of the problems was not significant. Each problem appeared on a separate sheet of paper along with other, irrelevant questions. Each sheet was taken away after the student had responded. Then, the exposure task was administered.

Results

Division of a Line Segment

As can be seen in Table 1, two types of responses were given to the problem related to the successive division of a line segment before the exposure to the appropriate answers: (a) the process is endless and (b) the process will come to an end.

The frequency of the adequate, infinite response significantly increased with grade level ($\overline{X} = 36.27$, df = 3, $p < .001$) and was relatively high in the upper grades (see Table 1).

Students used three types of justifications to the endless response. The most dominant one, in all grade levels, was, "One can always divide by two." The percentage of students who used this justification, which referred to the dynamic aspect of the process, increased with grade level. The second justification was, "There is an infinite number of points in a line segment." This type of justification was mainly given by students in the upper grades. It probably reflects the effects of the instruction in geometry which emphasizes that a line segment is composed of an infinite number of points. The third, less frequent, justification was, "We shall reach a point but a point can also be divided." This notion of a divisible point might reflect the existence of two contradictory ideas simultaneously held by these students: (a) the idea that the process of successive division by two can continue endlessly and (b) a conception of a line segment as an entity which is composed of a finite number of points. Students who used this hybrid notion described these two ideas in their answers. For instance, "The line segment consists of a finite number of points, but it is possible to divide a point into two."

Students used three types of justifications for their inadequate response that the process of halving the line segment will come to an end. A substantial number of seventh-grade students argued that, "We shall not be able to divide anymore because the segment will become extremely small." These students referred to the actual process of division and emphasized its technical limitations. The second justification was that, "The segment is finite." This justification was probably dominated by the idea that the segment is bounded. Few students in this category added that the segment contains a finite number of points. The third type of justification was, "We shall not be able to divide anymore as we shall reach the basic unit of the segment." Some of the students who used this justification (mainly the seventh-grade students who had just studied about the particulate nature of matter) referred to atoms while others (mainly the tenth- and twelfth- grade students who had received rather intensive instruction in mathematics referred to points.

Division of a Copper Wire

As can be seen in Table 2, two types of responses were given, before intervention, to the problem related to the halving of the copper wire: (a) the process will come to an end and (b) the process is endless.

In the case of the copper wire, as in the case of the line segment, the frequency of inadequate infinite responses significantly increased with grade level ($\overline{X} = 11.07$, df = 3, $p < .01$). This surprising increment in the percentage of these responses in the upper grade levels is probably due to the effect of the rather intensive instruction in mathematics these students received in the upper grades. This instruction, which emphasized the ideal nature of line segments, might encourage students from the upper grades to draw an inappropriate analogy between the line segment and the copper wire.

The justifications that students gave to their answers to the copper wire problem were similar to those given to the line

Table 1

The Division of a Line Segment

Grade	Before Intervention 7	8	10	12	After Intervention 7	8	10	12
The process is endless								
Total	26	50	78	78	28	54	80	86
One can always divide by two	24	40	48	58	24	30	38	56
There is an infinite number of points in a line segment	0	4	26	20	4	8	24	20
We shall reach a point, but it can also be divided	2	6	4	0	0	16	18	10
The process will come to an end								
Total	74	50	22	22	62	46	18	14
We shall not be able to divide anymore, because the segment will become extremely small	44	18	6	4	14	30	8	6
The segment is finite	8	22	2	6	6	8	4	2
We shall not be able to divide anymore as we shall reach the basic unit of the segment:								
a point	4	4	12	10	14	6	2	6
an atom	18	6	2	2	28	2	4	0
I do not know	0	0	0	0	10	0	2	0

Table 2

The Division of a Copper Wire

Grade	Before Intervention 7	8	10	12	After Intervention 7	8	10	12
The process will come to an end								
Total	76	74	50	50	74	88	78	66
We shall not be able to divide anymore, because the wire will become extremely small	36	28	22	26	26	28	22	30
There is a finute number of atoms in the copper wire	10	6	2	0	8	6	10	12
We shall not be able to divide anymore as we shall reach the basic unit of the wire:								
an atom	30	38	26	24	40	54	46	24
a point	0	2	0	0	0	0	0	0
The process is endless								
Total	24	26	50	50	16	12	20	34
One can always divide by two	20	22	42	40	10	6	8	20
The wire is infinite	0	2	0	0	2	2	0	0
We shall reach an atom, but it can also be divided	4	2	8	8	4	4	12	14
I do not know	0	0	0	0	10	0	2	0

segment problem. Again, three main justifications were givento the adequate, finite answers, "We shall not be able to divide anymore because the wire will become extremely small," "There is a finite number of atoms in the copper wire," and "We shall not be able to divide anymore as we shall reach a basic unit of the copper wire." Among the justifications given to the incorrect, infinite answers, the most common, at all grade levels, was, "One can always divide by two." Another justification, given by only a few students, was, "The wire is infinite." A third justification was, "We shall reach an atom but it can also be halved". This last, incorrect justification could evolve from integrating the idea that "One can always divide by two" with the particulate theory of matter. Some of these students inappropriately used their knowledge that an atom consists of elementary particles and referred to it as an entity that can be halved. For instance, one student argued that

"Everything can be divided by two but I studied in science that matter consists of a finite number of atoms. When we reach an atom, it will explode and thus the process of dividing by two will continue forever."

Thus far, the data show that the same answers and even the same justifications were provided by the students to both problems; however, these data do not provide any indication of the consistency of responses at the individual level. Such information is provided by examining the students' response patterns to the two problems.

Response Patterns to the Successive Division Problems

The four possible response patterns are presented in Table 3. Students who gave the same responses to both problems (either finite or infinite) are included under the concordant patterns. Those who gave different answers to the problems are grouped under the discordant patterns.

The discordant patterns. Table 3 shows that before intervention, the percentage of the students who showed discordant response patterns was relatively low. As expected, the most frequent pattern among the two discordant ones was the correct, infinite-finite response pattern. Although the frequency of the infinite-finite response significantly increased with grade level ($\overline{X} = 12.24$, df = 3, $p < .001$), even among the higher grade levels, no more than 36% of the students showed this adequate response pattern. Very few students at each grade level gave reversed answers to these problems---a finite answer to the segment problem and an infinite answer to the copper wire one.

The concordant patterns. Table 3 shows that most students, at all grade levels, gave the same response to both problems--either that the process of division is endless (the infinite-infinite pattern) or that the process will come to an end (the finite-finite pattern). Almost all of the seventh-grade students (86%) gave the same response to both problems. From the eighth grade onwards, the percentage of the students who gave the same response was between 56% and 68%.

The frequency of the infinite-infinite pattern significantly increased with grade level ($\overline{X} = 18.72$, df = 3, $p < .001$). This increase is probably due to the extensive instruction of mathematics in the upper grades which apparently influenced the students' responses not only with regard to the line segment problem but also to the copper wire one. The finite-finite pattern significantly decreased with grade level ($\overline{X} = 34.76$, df = 3, $p < .0001$). This decrease may indicate that students in the upper grade levels gave up their initial, adequate finite response to the copper wire problem in favor of the infinite response.

It is notable that almost all the students who showed a concordant response pattern also gave the same justification to both problems. Most students who showed the infinite-infinite pattern used the "One can always divide by two" justification. Students who showed the finite-finite pattern gave either the justification that "We shall not be able to divide anymore because the segment (or the wire) will become extremely small" or that "We shall not be able to divide anymore as we shall reach a basic unit (an atom of a point)." Some students explicitly referred in their answers to the apparent concordance of these problems. For instance, "The process of dividing the copper wire will come to an end, and this is exactly as with the segment, it is the same principle. It will happen when we reach the smallest part."

The Effects of the Intervention

Table 1 shows that in each grade level, the percentages of the correct responses which were given by the students before and after the intervention are similar. Mcnemar tests with p of

Table 3

Response Patterns to the Line Segment and to the Copper Wire Problems

		Before Intervention				After Intervention			
Grade		7	8	10	12	7	8	10	12
Segment	Wire								
Discordant Patterns									
Total		14	42	32	44	12	54	60	56
Infinite	Finite	8	34	30	36	12	48	60	54
Finite	Infinite	6	8	2	8	0	6	0	2
Concordant Patterns									
Total		86	58	68	56	78	46	38	44
Infinite	Infinite	18	16	48	42	16	6	20	32
Finite	Finite	68	42	20	14	62	40	18	12
I do not know		0	0	0	0	10	0	2	0

.01, which were carried out for each of the grade levels, did not indicate significant differences in any of the grade levels. Thus, it seems that the effect of exposure to the appropriate answers on students' responses to the line segment problems was minor.

Although the majority of students did not change their responses to the line segment problem, many changed their justifications to both the finite and infinite responses after the intervention. With respect to the infinite justifications, it is apparent that after the intervention, the frequency of the justification that "One can always divide by two" decreased in grades eight and ten while the idea that "We can reach a point but a point can also be divided" increased in grades eight, ten, and twelve. With regard to the finite justifications, it was evident that in the seventh grade, the frequency of the justification "We shall not be able to divide anymore because the segment will become extremely small" decreased from 44% to 14%, while the particulate explanation increased from 22% to 42%. This may be due to the fact that seventh-grade students had, during that year, studied the particulate nature of matter.

As can be seen in Table 2, the effect of the intervention on students' responses to the copper wire problems differs from its effect on their responses to the line segment problems. After the intervention, the percentages of the finite, correct responses to the copper wire problem significantly increased in all grades except for the seventh (Mcnemar tests indicate significant differences at the levels of $p < .0.01$, $p < .001$, and $p < .01$ for grades eight, ten, and twelve, respectively). When examining the students' justifications to their finite, adequate response, it became apparent that the percentage of students who explicitly referred to the notion of the atom increased at all grade levels. In respect to students' justifications of their infinite, inadequate response, there is a decrease at all grade levels in the number of students who argued that "One can always divide by two." Seventh-grade students who had abandoned this justification are those who admitted that they did not know the answer to the problem. Most of the students in the eighth, tenth, and twelfth grades who gave up this justification resorted to a finite response and referred to the atomic level. Others, who remained faithful to the infinite response, also referred to the atomic level and argued that "We shall reach an atom but it can also be divided."

After the exposure to the correct answers, in each grade level, less students exhibited the erroneous concordant response patterns, and more students exhibited the discordant, adequate, infinite-finite response pattern (see Table 3). The percentages of students in the tenth and twelfth grades who exhibited the correct, infinite-finite response pattern after the intervention were significantly higher than those before the intervention (Mcnemar tests indicate significant differences at the levels of $p < .001$ and $p < .01$ for grades ten and twelve, respectively and no significant differences for grades seven and eight). The discordant, finite, infinite reversed pattern almost disappeared.

It is noteworthy that many of the students who correctly responded to both problems after the intervention (but not before it) commented on the apparent differences between the two problems. For instance, one student argued that "Karen is right because it is impossible to half an atom but the division of the line segment is infinite because there is an infinite number of points in the line segment."

Discussion

The first task in this study was to examine the students' responses to two externally similar but essentially different problems. One of these problems asked the students to determine whether the halving of a line segment will have an end, whereas the other problem exposed them to a similar process of halving, yet with reference to a copper wire. The data indicate that the majority of the students gave the same response to both problems--either a finite response, especially with the younger ones, or an infinite response, especially among the older ones.

These findings beg the following questions: What caused the students to give the same response to these essentially different problems? Why did most of the younger students use the finite-finite response pattern whereas the older ones tended to show the infinite-infinite pattern?

Studies which examined differences in scientific problem solving between novice and expert solvers revealed that inexperienced solvers tend to mentally represent a given problem according to surface features whereas experienced solvers refer to the scientific (or mathematical) concepts and principles (Chi, Fletovich, & Glaser, 1981; Larkin & Rainard, 1984). This may suggest that many of the subjects, who can be regarded as inexperienced problem solvers in both science and mathematics, responded in the same manner to both problems because they formed mental representations which were based on the external, figural similarity of the entities in the two problems and on the apparent identity of the process involved. The fact that some of the subjects explicitly referred to the resemblance between the problems further supports the assumption that the external resemblance between the problems acted as a trigger which led the students to give the same answers to both problems.

The previous paragraph suggests an explanation to the finding that the same response was provided by the students to both different problems; however, there is still the need to explain why most of the younger students exhibited the finite-finite response pattern while most of the older ones showed the infinite-infinite response pattern. In considering this issue, it is important to recall that studies investigating how children and adolescents cope with the concept of infinity have found that students gave two different, intuitive answers to problems that dealt with infinite processes--a finite response and an infinite response (Fischbein, Tirosh, & Melamed, 1981; Piaget & Inhelder, 1963; Tall, 1981). Further, in problems that dealt with successive division of line segments, a tendency towards domination of the infinite answers in the higher grades was observed (Duval, 1983; Fischbein, Tirosh, & Hess, 1979). This increase was explained in terms of the effects of the formal, school-based mathematical instruction that dealt with infinite processes. The findings in this study are compatible with those of the studies on students' understanding of infinity and can also be interpreted in light of the effect of formal, school-based instruction. The younger students, who had studied the particulate nature of matter but not the ideal, abstract nature of a line segment, tended to give finite responses to both questions. Instruction in the particulate nature of matter, which they received in science, served as a source of support to the intuitive, finite response pattern. Similarly, the older students, who had been exposed to rather intensive mathematical instruction on geometrical concepts and infinite processes, tended to produce the infinite response. The mathematical instruction they received supported the intuitive, infinite response pattern.

What is particularly interesting is that many students in tenth and twelfth grades, who had, in fact, acquired the necessary formal knowledge to correctly respond to each of these problems, abandoned their finite, adequate response to the copper wire problem in favor of the infinite one. This surprising finding may stem from the fact that the older subjects are mathematics majors. It may very well be that if other students were tested (for example, science majors), they would have yielded other response patterns. This is still under investigation.

The results also indicate that a relatively minor intervention, which only exposed students to the appropriate responses to each of the problems, was very effective for students in the upper grades. After the intervention, many of these students limited the use of infinite model only to the line segment problem. It is most probable that this type of intervention can be influential only for students who hold the adequate, formal school-based knowledge in both science and mathematics since they can safely lean on this knowledge when confronted with the adequate responses. The intervention impelled the older students to disregard the external similarity and to focus on the qualitative differences between the problems. Consequently, some of these students were able to form different, adequate mental representations to each of these problems.

This last statement leads to the main implications that can be drawn from this study. For researchers who are studying the nature of students' responses in one specific content domain, it is imperative to consider the possible influences of knowledge acquired in related domains on students' responses to problems embedded in the objective domain. Teachers should be aware of students' natural tendency to relate to surface features of given problems and thus should attempt to help students set the limits for application of their newly acquired knowledge to a specific domain.

This study, as well as others, has suggested that the mental representation of the problem has a crucial effect on students' responses (Behr, Reiss, Harel, Post, & Lesh, 1986; Chi, Glaser & Rees, 1982; Kaput, 1987; Larkin, 1983). Furthermore, the representations of inexperienced problem solvers greatly lean on external, nonessential, and not necessarily scientific features of the problem. This implies that if one intends to help learners improve their ability to solve scientific problems, one should devote many more efforts to develop ways to assist students in forming mental representations that consider the scientific nature of the problem.

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